PROJECTION METHODS FOR THE CALCULATION OF INCOMPRESSIBLE UNSTEADY FLOWS

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A general formula for the second-order projection method for solution of unsteady incompressible Navier-Stokes equations is presented. It includes the four- and three-step projection methods. Also, RKCN (Runge-Kutta/Crank-Nicholson) three-step and four-step projection methods are presented, in which the three-stage Runge-Kutta and semi-implicit Crank-Nicholson techniques are employed to update the convective and diffusion terms, respectively. The RKCN projection method is further simplified. The pressure Poisson equation (PPE) is solved only at the final substage for the simplified RKCN projection method, which greatly reduces the computation time. The high-order boundary conditions for the intermediate velocities have also been given for the four-step RKCN projection method and its simplified version. A 2-D vortex flow, a 2-D oscillating cavity flow, and a 3-D lid-driven cavity flow are simulated to validate the analysis. The projection method is also used to do the direct numerical simulation (DNS) of a fully developed channel flow.

1. INTRODUCTION

The dimensionless unsteady incompressible Navier-Stokes equations in primitive variables can be written as

\[ \frac{\partial u}{\partial t} + u \cdot \nabla u = -\nabla p + \frac{1}{\text{Re}} \nabla^2 u \]  

\[ \nabla \cdot u = 0 \]  

Received 9 December 2002; accepted 16 May 2003.
Ming-Jiu Ni acknowledges financial support from the Japan Society for the Promotion of Science (JSPS).
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where \( u, p \) are the nondimensional velocity vector and kinetic pressure, respectively, and \( \text{Re} \) is the Reynolds number.

It is a central issue to develop an appropriate discrete form of the incompressibility constraint for the design of numerical methods for Navier-Stokes (N-S) equations. The famous primitive-variable numerical methods include the MAC method [9], the projection method [1, 3–7, 12], the SIMPLE method [21, 22], and others. The MAC method [9] is an explicit transient algorithm. The time-step size is restricted to be proportional to the square of the spatial step size. In [14], the MAC method is employed to do the calculation of interfacial flows. The SIMPLE algorithm of Patankar and Spalding [22] employs implicit updating techniques for both convective and diffusion terms. The time step can be greatly improved. Chorin [5] developed the projection method or fractional step method, in which the diffusion term is implicitly updated and the time-step size is also greatly enlarged. For the unsteady incompressible Navier-Stokes equation, Gresho [7] presented a detailed discussion of the projection method, which contains methods of first, second, and third-order temporal accuracy. The optimal boundary conditions for intermediate velocities have been given. It was also shown that the simplified projection methods and the simplified boundary conditions can give the same accuracy solution as the optimal one beyond a very thin boundary layer. Dukowicz and Dvinsky [6] designed a general high-order projection method based on the application of the approximation factorization (AF) technique on the Navier-Stokes system. By employing block LU (lower and upper triangular matrices) decomposition, Perot [24, 25] analyzed the solution accuracy of the projection method and the influence of boundary conditions on it.

After a brief review of the projection methods, we present a general second-order formula of the projection method for the incompressible N-S equation in Section 2. The general formula is further formed as four-step and three-step projection methods. In Section 3, the RKCN four- and three-step projection methods are developed. The simplified version of the three-stage RKCN four-step projection method is also presented, in which the PPE needs to be solved only at the last substage. The higher-order boundary conditions for the intermediate velocities are also given. In Section 4, 2-D vortex flows and oscillating cavity flows are simulated by the RKCN four-step projection method and its simplified version to validate the analysis. Also, 3-D lid-driven cavity flows for \( \text{Re} = 400 \), and \( \text{Re} = 3,200 \) are conducted. The steady and unsteady vortexes show that the computation method has high accuracy. Direct numerical simulation (DNS) of turbulence with nonuniform meshes was performed for a fully developed turbulent channel flow. The conclusions are presented in Section 5.

### 2. GENERAL SECOND-ORDER PROJECTION FORMULA

The straightforward temporal discretization of Eqs. (1a) and (1b) can be written as

\[
\frac{v^{n+1} - v^n}{\Delta t} + \alpha \nabla p^{n+1} = \left( \frac{\nabla^2 v}{\text{Re}} \right)^{n+1/2} - (v \cdot \nabla v)^{n+1} - (1 - \alpha) \nabla p^n \quad (2a)
\]
Here the backward Euler technique is employed for the updating of the time derivative term. \( v \) and \( p \) are the unknown discrete velocity vector and pressure, \((\nabla^2 v/Re)^{n+1/2}\) and \(- (v \cdot \nabla) v^{n+1/2}\) are the temporal updating of the diffusion and convective terms, respectively. The pressure term is updated by \( \alpha \nabla p^{n+1} + (1 - \alpha) \nabla p^n \cdot \alpha \) is a coefficient constant. We have trapezoidal updating of the pressure term when \( \alpha = \frac{1}{2} \), and fully implicit updating when \( \alpha = 1 \). The solution of Eqs. (2a) and (2b) is not easy to obtain, because it involves simultaneous solution for the velocity and pressure. Projection methods first solve the convection-diffusion equation to predictor intermediate velocities, which are then projected onto the space of divergence-free field.

Following the matrix analysis method [6, 24], the discretized Eqs. (2) can be written as the matrix form:

\[
\begin{pmatrix}
A & G \\
D & 0
\end{pmatrix}
\begin{pmatrix}
v^{n+1} \\
\alpha p^{n+1}
\end{pmatrix}
= \begin{pmatrix}
r - (1 - \alpha)Gp^n \\
0
\end{pmatrix}
\]

where \( A, G, \) and \( D \) are submatrices, the right-hand-side \( r \) vector contains all those quantities that are already known. Different updating techniques for the convective and diffusion terms will produce different formulations of \( A \) or \( r \). Considering the explicit updating of the convective term for simplicity and semi-implicit Crank-Nicholson updating of the diffusion term for stability, the discretized Navier-Stokes equations (2) can be written as the matrix form (3) with

\[
A = \frac{1}{\Delta t} \left( I - \frac{\Delta t}{2Re}L \right) \\
r = \frac{1}{\Delta t} \left( I + \frac{\Delta t}{2Re}L \right) v^n - N^{n+1/2}(v)
\]

where \( N^{n+1/2}(v) \) is the discrete convective operator with second-order temporal accuracy, \( I \) is the unit identity matrix operator, \( L \) is the discrete Laplacian operator, \( D \) is the discrete divergence operator, and \( G \) is the discrete gradient operator.

The block matrix form of system (3) can be factored as

\[
\begin{pmatrix}
I & 0 \\
DA^{-1} & I
\end{pmatrix}
\begin{pmatrix}
A & G \\
0 & -DA^{-1}G
\end{pmatrix}
\begin{pmatrix}
v^{n+1} \\
\alpha p^{n+1}
\end{pmatrix}
= \begin{pmatrix}
r - (1 - \alpha)Gp^n \\
0
\end{pmatrix}
\]

or

\[
\begin{pmatrix}
A & 0 \\
D & -DA^{-1}G
\end{pmatrix}
\begin{pmatrix}
I & A^{-1}G \\
0 & I
\end{pmatrix}
\begin{pmatrix}
v^{n+1} \\
\alpha p^{n+1}
\end{pmatrix}
= \begin{pmatrix}
r - (1 - \alpha)Gp^n \\
0
\end{pmatrix}
\]

The latter can be reduced to the following set of equations:

\[
A\tilde{v} = r - (1 - \alpha)Gp^n
\]

\[
\alpha DA^{-1}Gp^{n+1} = D\tilde{v}
\]

\[
v^{n+1} = \tilde{v} - \alpha A^{-1}Gp^{n+1}
\]
which is referred to as the Uzawa method. However, the Uzawa method is extremely expensive computationally, since matrix $A$ must effectively be inverted for every iteration of the discrete pressure Poisson equation. The fractional step method approximates Eq. (3) and significantly reduces the computational complexity by assuming that $A^{-1} = \Delta t I$ as

$$
\begin{pmatrix}
A & (\Delta t A)G \\
D & 0
\end{pmatrix}
\begin{pmatrix}
v^{n+1} \\
\alpha p^{n+1}
\end{pmatrix} =
\begin{pmatrix}
r - (1 - \alpha) G p^n \\
0
\end{pmatrix}
$$

(7)

The pressure gradient terms in the momentum equations have been altered. The approximation has only first-order temporal accuracy with an error term as $a \Delta t I = (\Delta t A^{-1}) G p^{n+1}$, which will be $\alpha [(\Delta t/2 \text{Re})] LG p^{n+1}$ if the Crank-Nicholson scheme is used to update the diffusion term. This method was first used to construct the projection method in [5].

According to the error analysis result of the first-order approximation (7), here we consider the following approximation to system (3):

$$
\begin{pmatrix}
A & (\Delta t A)G \\
D & 0
\end{pmatrix}
\begin{pmatrix}
v^{n+1} \\
\alpha p^{n+1}
\end{pmatrix} =
\begin{pmatrix}
r - (1 - \alpha) G p^n - (\alpha I - \Delta t A) G p^n \\
0
\end{pmatrix} =
\begin{pmatrix}
r + (\alpha \Delta t A - I) G p^n \\
0
\end{pmatrix}
$$

(8)

where the right-hand-side vector has been augmented by an extra term. Equation (8) is a second-order approximation to system (3) with error term $\alpha \Delta t [(\Delta t I/\Delta t - A)] G [(p^{n+1} - p^n)/\Delta t]$, which will be equal to $\alpha [(\Delta t)^2/2 \text{Re})] LG [(p^{n+1} - p^n)/\Delta t]$ when the diffusion term is updated using the Crank-Nicholson scheme.

The coefficient matrix of Eq. (8) can be factored by block decomposition as

$$
\begin{pmatrix}
A & 0 \\
D & -(\Delta t D)G
\end{pmatrix}
\begin{pmatrix}
\tilde{v} \\
\alpha p^{n+1}
\end{pmatrix} =
\begin{pmatrix}
r + (\alpha \Delta t A - I) G p^n \\
0
\end{pmatrix}
$$

(9a)

$$
\begin{pmatrix}
I & \Delta t G \\
0 & I
\end{pmatrix}
\begin{pmatrix}
v^{n+1} \\
\alpha p^{n+1}
\end{pmatrix} =
\begin{pmatrix}
\tilde{v} \\
\alpha p^{n+1}
\end{pmatrix}
$$

(9b)

Equations (9a) and (9b) can be written in the series of operation as

$$
A \tilde{v} = r + (\alpha \Delta t A - I) G p^n
$$

(10a)

$$
\alpha \Delta t DG p^{n+1} = D \tilde{v}
$$

(10b)

$$
v^{n+1} = \tilde{v} - \alpha \Delta t G p^{n+1}
$$

(10c)

The projection method has second-order temporal accuracy, and (10a) can be reformulated as

$$
A(\tilde{v} - \alpha \Delta t G p^n) = r - G p^n
$$

(11)

Letting $\hat{v} = \tilde{v} - \alpha \Delta t G p^n$, we have the following four-step projection method for incompressible flows:

$$
A \hat{v} = r - G p^n
$$

(12a)
\[ \ddot{v} = \dot{v} + \alpha \Delta t \, G \, p^n \]  
\[ \alpha \Delta t \, DG \, p^{n+1} = D\ddot{v} \]  
\[ v^{n+1} = \ddot{v} - \alpha \Delta t \, G \, p^{n+1} \]

The above equations can also be formulated as a three-step projection step method:

\[ A\ddot{v} = r - G \, p^n \]
\[ \alpha \Delta t \, DG \left( p^{n+1} - p^n \right) = D\ddot{v} \]
\[ v^{n+1} = \ddot{v} - \alpha \Delta t \, G \left( p^{n+1} - p^n \right) \]

The general four-step projection method (12) will form the Choi and Moin method [4] with \( \alpha = 1 \). In [4], the semi-implicit second-order Crank-Nicholson scheme is employed for updating of both convective and diffusion terms, to improve the stability. It should be noted that the implicit updating of the convective term does not favor the implementation of higher-order spatial schemes. The second-order central difference (CD) scheme is utilized to conduct the spatial discretization of the convective term in [4].

The second-order-accuracy Dukowicz and Dvinsky method [6] is often employed to do the DNS of turbulent flows. It needs to solve the Poisson equation of a pressure difference. In fact, the general three-step projection method of (13) with \( \alpha = \frac{1}{2} \) forms the method in [6]. The original second-order method of [6] was designed for solving the Stokes flow, in which the Crank-Nicholson scheme was employed to update the diffusion term with good stability. The projection method (13) with \( \alpha = 1 \) was also employed to simulate boiling heat transfer [29] incorporating the level set approach, in which the first-order fully implicit scheme and explicit scheme were employed to update the diffusion and convective terms, respectively.

Bell, Colella, and Glaz [1] developed a projection method in which Crank-Nicholson is employed for updating of the diffusion term and Godunov is used for updating of the convective term. Usually, the Godunov scheme has only second-order spatial accuracy. This projection method is very robust, and the variable density version of this projection method [2] has been successfully applied to the simulation of unsteady interfacial flows incorporating the volume-of-fluid (VOF) [26] and level set [30] approaches. This projection method is also a special case of the general three-step projection method with \( \alpha = 1 \).

The standard SIMPLE algorithm [21, 22] is formulated to solve steady flows, not for solving transient flows. The fundamental concept of the SIMPLE method is to derive a pressure-correction equation by enforcing mass continuity over each cell. Here we rewrite the SIMPLE method as fractional formulations:

\[ \left( \frac{1}{\Delta t} + \frac{A^n_p}{\Delta t} \right) \ddot{v}_p = \frac{v^n_p}{\Delta t} + \sum A^n_m \dot{v}_M - G \, p^n \]
\[ D \left[ \frac{\Delta t}{1 + \frac{A^n_p}{\Delta t}} G(p^{n+1} - p^n) \right] = D\ddot{v}_p \]
In Eqs. (14), the $A$ coefficient terms contain the contribution of the convective and diffusion terms. The index $M$ is a grid identifier referring to all the nodes surrounding the pole nodes that are involved in the formulation of the finite-difference representation of the spatial fluxes. The subscript $P$ denotes the pole node. Similar to Patankar [22], linear treatment of the convective term is applied in the above discretization. For the SIMPLE method, the Poisson equation (14b) will be solved to get the pressure difference, which will be utilized to correct the velocity by (14c). If $A_P^n$ is set to zero in Eqs. (14b) and (14c), the SIMPLE method is in fact the three-step projection method as described in Eq. (13) without considering the different updating techniques for the convective and diffusion terms in momentum equations. However, due to the linear treatment for the nonlinear term, the iteration is needed for the SIMPLE method to solve unsteady flows at every time level.

3. RKCN PROJECTION METHODS

A simple Fourier stability analysis of the Adams-Bashforth method for the convective term as applied to the 1-D linear hyperbolic equation shows that this method is unstable for all Courant, Freidricks, and Levy (CFL) numbers. The implicit techniques for the convective term in Navier-Stokes equations will make it difficult to solve the discretized algorithm equations, especially for the higher-order spatial discretization schemes. The Runge-Kutta method seems more suitable for the convective term because of its stability and simplicity. Rai and Moin [27] employed an explicit low-storage Runge-Kutta method, which has the additional advantage that it requires the minimum amount of computer runtime memory for the convective term and the Crank-Nicholson technique for the diffusion term. The overall accuracy of the method is second-order in time. However, they employed Kim and Moin’s projection method [12], which has reduced accuracy for most boundary conditions [24]. The method has been extended to a semistaggered grid system [10]. Now we present a RKCN projection method based on the general second-order formula of the projection method. The three-stage RKCN four-step projection method be expressed as

$$A^m(v^m - v^{m-1}) = \left( \frac{2v^m}{Re} \right) L(v^{m-1}) + r_N^m - G(\alpha^m p^{m-1} + \beta^m p^{m-2})$$  \hspace{1cm} (15a)

$$\tilde{v}^m = v^m + \alpha \Delta t G(\alpha^m p^{m-1} + \beta^m p^{m-2})$$  \hspace{1cm} (15b)

$$v^m = \tilde{v}^m - \alpha \Delta t G(\alpha^m p^m + \beta^m p^{m-1})$$  \hspace{1cm} (15c)

$$\alpha \Delta t DG(\alpha^m p^m) = D\tilde{v}^m - \alpha \Delta t DG(\beta^m p^{m-1})$$  \hspace{1cm} (15d)

where

$$\alpha^m = \begin{pmatrix} 8 & 5 & 3 \\ 15 & 12 & 4 \end{pmatrix} \quad \beta^m = \begin{pmatrix} 0 & 17 & -5 \\ 60 & 12 \end{pmatrix} \quad \gamma^m = \begin{pmatrix} 4 & 1 & 1 \\ 15 & 15 & 6 \end{pmatrix}$$

$$A^m = \frac{1}{\Delta t} \left( I - \frac{\Delta t v^m}{L} \right) \quad r_N^m = -\alpha^m N(v^{m-1}) - \beta^m N(v^{m-2})$$
Here \( N(v) \) represents the discretization formula of the nonlinear term. The divergence form, skew-symmetry form, or convective form can be employed to discretize the convective term. Here, to keep the conservativeness of mass, kinetic energy, and momentum, the skew-symmetry form is employed to conduct the spatial discretization. The velocity components and pressure in the intermediate velocities equation at the first substage as:

\[
v^n_{/C0} = 0, \quad p^n_{/C0} = 0 \quad (m = 2 = -1) \quad \text{and} \quad v^0 = v^n, \quad p^0 = p^n (m = 1 = 0).
\]

At the third stage \( v^3 = v^{n+1} \), \( p^3 = p^{n+1} \), which are the updated velocities and pressure for the next time level.

The above RKCN techniques can also be employed to update the convective and diffusion terms, respectively, for the three-step projection method (13) as follows:

\[
A^m (\tilde{v}^m - v^{m-1}) = \left( \frac{2\xi^m}{\text{Re}} \right) L(v^{m-1}) + r^m - G(\alpha^m p^{m-1} + \beta^m p^{m-2}) \tag{16a}
\]

\[
v^m = \tilde{v}^m - \alpha \Delta t G[\alpha^m (p^m - p^{m-1}) + \beta^m (p^{m-1} - p^{m-2})] \tag{16b}
\]

\[
\alpha \Delta t DG[\alpha^m (p^m - p^{m-1})] = D\tilde{v}^m - \alpha \Delta t DG[\beta^m (p^{m-1} - p^{m-2})] \tag{16c}
\]

For the three-stage RKCN three-step projection method, the Poisson equation (16c) will be solved to get the pressure difference, which will be further used to correct the velocity by (16b). Both the RKCN three-step and four-step projection methods need to solve Poisson equations for the pressure or pressure difference in every stage, which will consume much more computational time, although we can employ multigrid and Krylov methods to accelerate the convergence. In [15], the RM method has been simplified. In the simplified version of RM method, the pressure Poisson equation needs to be solved only at the last substage, and it will save computation time. It has been stressed in [15] that the simplified version also has second-order accuracy, like the original one. Here we present a simplified version of the three-stage RKCN (SRKCN) four-step projection method (15) as

\[
A^m (\tilde{v}^m - v^{m-1}) = -G[(\alpha^m + \beta^m) p^m] + r^m_{/N} - \left( \frac{2\xi^m}{\text{Re}} \right) L(v^{m-1}) \tag{17a}
\]

\[
\tilde{v}^m = \tilde{v}^m - \alpha \Delta t G[(\alpha^m + \beta^m) p^m] \quad (m = 1, 2, 3) \tag{17b}
\]

\[
v^m = \tilde{v}^m - \alpha \Delta t \sum_{i=1}^m (\alpha^i + \beta^i) G p^m \quad (m = 1, 2) \tag{17c}
\]

\[
v^{n+1} = \tilde{v}^3 - \alpha \Delta t G p^{n+1} \tag{17d}
\]

\[
\alpha \Delta t DG p^{n+1} = D\tilde{v}^3 \tag{17e}
\]

The RKCN four-step projection method and its simplified version have also been extended to staggered grid [9] and collocated grid systems [28, 31]. Higher-order spatial discretization schemes, such as spectral-like compact schemes [16] and fully conservative high-order schemes [19] can be easily incorporated with the above

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RKCN projection methods. A spatial high-order accurate scheme is in favour of the resolution of small scales of turbulence in DNS and LES [18]. For projection methods, it is important to present high-order-accuracy boundary conditions for the intermediate velocities. For the three-stage RKCN four-step projection method and its simplified version, we have the following.

**Boundary Conditions**

For the three-stage RKCN four-step projection method, the second-order temporal boundary conditions for the intermediate velocities can be given as

\[ \hat{v}^m - \hat{v}^{m-1} = (\alpha^m + \beta^m)(v^{n+1} - v^n) \]  

(18a)

\[ v^m = v^n + \sum_{l=1}^{m} (\alpha^l + \beta^l) (v^{n+1} - v^n) \]  

(18b)

For the simplified RKCN four-step projection method, we present the following two kinds of boundary conditions for the intermediate velocities.

**BCs-A:**

\[ \hat{v}^m - \hat{v}^{m-1} = (\alpha^m + \beta^m)(v^{n+1} - v^n) \]  

(19a)

\[ \hat{v}^m = v^n + \sum_{l=1}^{m} (\alpha^l + \beta^l) (v^{n+1} - v^n) \]  

(19b)

\[ v^m = v^n + \sum_{l=1}^{m} (\alpha^l + \beta^l) (v^{n+1} - v^n) \]  

(19c)

**BCs-B:**

\[ \hat{v}^m - \hat{v}^{m-1} = (\alpha^m + \beta^m)(v^{n+1} - v^n) \]  

(20a)

\[ \hat{v}^m = v^n + \sum_{l=1}^{m} (\alpha^l + \beta^l) [(v^{n+1} - v^n) + \Delta t \alpha Gp^n] \]  

(20b)

\[ v^m = v^n + \sum_{l=1}^{m} (\alpha^l + \beta^l) (v^{n+1} - v^n) \]  

(20c)

BCs-A has only first-order accuracy for the velocities, while BCs-B has second-order accuracy for the velocities. The computation comparison for the BCs-A and BCs-B will be conducted in the next section.

### 4. NUMERICAL VALIDATION

#### 2-D Vortex Flows

The proposed schemes and boundary conditions for the RKCN four-step projection method and its simplified version are tested in computing the following 2-D unsteady flow of a decaying vortex, which has the following exact solution:

\[ u(x, y, t) = -\cos(x) \sin(y) e^{-2t} \]  

(21a)
This example has been simulated in [12, 15] to validate the temporal accuracy of a projection method. Computations are carried out in the domain $0 \leq x, y \leq \pi$. In our computation, the uniform collocated meshes were used in the computation. On the boundaries, the exact solution was imposed. The same CFL number is used for all of the tested algorithms, which means the time-step size is proportional to the grid size. Figure 1 plots the maximum errors in $u$, at different time levels as a function of mesh refinement. The $y$ axis represents the maximum errors in $u$, while the $x$ axis represents the mesh points. Similar results were obtained for $v$. Figure 1 shows that the RKCN projection method is of second-order temporal accuracy, since the slope is greater than 2.

2-D Oscillating Lid-Driven Cavity Flow

Lid-driven flows in a 2-D square cavity at Reynolds number 1,000 and 1 (based on the lid velocity and cavity length) were simulated. The velocity boundary conditions are set to zero at all solid walls, except for the lid velocity, which is set to $u_0 = \sin(\pi t)$. The results are produced by the three-stage RKCN four-step projection method and its simplified version with $61 \times 61$ uniform collocated meshes and $10^{-1}$, $5 \times 10^{-2}$, $10^{-2}$, and $10^{-3}$ time-step sizes. This validation is used to check the
effect of boundary conditions of intermediate velocity. Our results using RKCN and SRKCN for \( \text{Re} = 1,000 \) are fully consistent with the results from Mohamad [17], in which the SIMPLER method with 121 \( \times \) 121 nonuniform meshes and \( 2 \times 10^{-4} \) time-step size are employed to get the benchmark solution. Due to space limitations, it is not shown here.

For the case of low Reynolds number with \( \text{Re} = 1 \), Figure 2 shows that the simplified version with first-order boundary conditions BCs-A (19) can get accurate results only when time step is less than or equal to 0.001, while SRKCN with BCs-B (20) can get accurate results even if the time step is 0.01. The numerical results show that the high-order-accuracy boundary conditions for the intermediate velocities play a great role for getting good numerical results and the high-order boundary conditions developed in this article can be applied to get high-accuracy numerical results for unsteady flows.

### 3-D Lid-Driven Flows in a Square Cavity

The lid-driven flow in a cubic cavity \((1 \times 1 \times 1)\) has been widely used for validation and comparison purpose. In the present work, we do the computation for \( \text{Re} = 400 \) and \( \text{Re} = 3,200 \), for which steady and unsteady transversal vortices are expected. The results for \( \text{Re} = 400 \) are shown in Figure 3, where the velocity profiles of the \( u \) component on the vertical centerline and the \( v \) component on the horizontal centerline of the plane \( z = 0.5 \) are shown. Both the positions and values of the extremes velocity are in a good agreement with Figure 3 in [10]. The velocity vector

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**Figure 2.** Temporal variations of \( u \) velocity at \( x = 0.5 \) and \( y = 0.25, \ 0.5, \ 0.75 \) for \( \text{Re} = 1 \) by SRKCN projection method with BCs-A and BCs-B.
Figure 3. 3-D lid-driven cavity flow for Re = 400: (a) $u-v$ vector plot at midplane of $z = 0.5$; (b) $v-w$ vector plot at midplane of $x = 0.5$; (c) $u-w$ vector plot at midplane of $y = 0.5$; (d) velocity profiles ($u$ component on vertical centerline, $v$ component on horizontal centerline).
Figure 3. Continued.
plots projected onto three orthogonal midplanes are also displayed in Figure 3. The plots in $x$–$z$ (b) and $y$–$z$ (c) clearly demonstrate that the flow is completely 3-D even at this low Re. The flow structure of the velocity field and the position of the transversal vortex core of present calculation are quantitatively consistent with the plots of Figure 4 of [10]. The velocity–vector plot of Figure 3 also clearly demonstrates the formation of steady transversal vortices, which are present in both the $x$–$z$ and $y$–$z$ planes. The strength of these transversal rolls is small compared with that of the main one.

Lid-driven flows in a 3-D cubic cavity at high Reynolds numbers have been investigated both experimentally and numerically. The highly unsteady flows exhibit complex flow structures such as the Taylor-Goertler-like (TGL) vortices first observed by Koseff et al. [13]. This flow serves as an excellent test case for time-accurate numerical methods. We computed the starting flow at a Reynolds number of 3,200. Numerical experiments by Perng [23] have shown that at this Reynolds number the flow is essentially symmetric over the midplane of the cavity.

In Figures 4 and 5, instantaneous streamlines and vortexes in the midplanes of $x = 0.5$ and $y = 0.5$ have been shown at $t = 20$ and $t = 70$. Strong unsteady phenomena can be seen. The complicated transversal vortex in the midplane of $y = 0.5$ shows that the flow is very complicated. At $t = 70$, we can clearly see the formation of the TGL vortex from the $v$–$w$ vector plot at the midplane of $x = 0.5$. The TGL vortex is a longitude vortex, which will contribute to the turbulence generation for high Reynolds number.

**DNS of Fully Developed Turbulence**

A direct numerical simulation (DNS) of the fully developed thermal field in a fully developed turbulent channel flow of air was carried out by the above-developed RKCN projection method. Periodic boundary conditions are applied in the streamwise and spanwise directions, while nonslip wall conditions are used for the wall-normal direction. A fourth-order, fully conservative central difference scheme [19] was employed to discretize the convective term. The fast Fourier transform (FFT) method is employed to solve the Pressure Poisson Equation (PPE). For this computation, the turbulent Reynolds number, based on wall friction velocity $u_t$ and width of channel $d$, was set at 150. The computational periods were chosen to be $2.5\pi\delta$ and $\pi\delta$ in the streamwise and spanwise directions, respectively. A staggered grid system is employed to avoid the checkerboard phenomenon. Uniform meshes with spacing $\Delta x^+ = u_t\Delta x/v = 18.4$, $\Delta z^+ = u_t\Delta z/v = 7.36$ are used in the streamwise and spanwise directions, nonuniform meshes of 128 points with hyperbolic tangent distribution are used with $\Delta y^+ = u_t\Delta y/v = 0.50 - 7.51$ in the wall-normal direction.

In this article, $(\ )^+$ means the normalization by the wall variables, $u_t$ (friction velocity), $v$ (kinetic viscosity), and $T_f$ (friction temperature).

The dimensionless mean velocity and temperature profiles are shown as a function of $y^+ = u_t y/v$ in Figure 6a. The root-mean-square velocity and temperature fluctuation normalized by the friction velocity and friction temperature are illustrated in Figure 6b. Figures 6c and 6d show the budgets of turbulent kinetic energy and temperature variance. From Figure 6, we can see all of the results agree closely with the results of Kasagi et al. [11], which were acquired by the high-accuracy
Figure 4. $v-w$ vector at midplane of $x = 0.5$ for 3-D lid-driven cavity flow at $Re = 3,200$: (a) $t = 20$; (b) $t = 70$. 

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Figure 5. $u-w$ vector at midplane of $y=0.5$ for 3-D lid-driven cavity flow at Re = 3,200: (a) $t=20$; (b) $t=70$. 
spectral method. This illustrates that the RKCN projection method can be applied on nonuniform meshes to get high-accuracy results. By employing a high-order spatial difference scheme, it can be used to do the DNS, LES research of the turbulence.
5. SUMMARY

Based on the matrix analysis method, a general second-order temporal accuracy formula of the projection method has been presented for unsteady incompressible N-S equations, which is further formed as the four- and three-step projection methods. Some existing well-known projection methods (or fractional-step
methods) have been analyzed. The SIMPLE method for unsteady flows has also been discussed, in which it has been written as the three-step formulas.

The RKCN multistep projection methods have been developed. The three-stage, low-storage Runge-Kutta technique is employed to explicitly update the convective term for stability and simplicity, and the semi-implicit Crank-Nicholson technique is employed to update the diffusion term for stability. The simplified version of the RKCN projection method also has been presented based on the four-step formula. The PPE needs to be solved only at the last substage to enforce the divergence-free velocities for the simplified version. It will save considerable computation time compared with the original one.

The second-order boundary conditions for intermediate velocities have been presented for the Runge-Kutta projection method and its simplified version. The numerical results show that high-accuracy boundary conditions for intermediate velocities play a big role in the acquisition of high-accuracy results.

Numerical examples of 2-D vortex flows and lid-driven cavity flows have been simulated to validate the analysis and conclusions in this article. The DNS result for a fully developed channel turbulent flow shows that the RKCN projection method can be used to accurately simulate a complex flow.

The present numerical methods have been further extended to the variable-density Navier-Stokes equations. A level set approach has been incorporated to simulate the 2-D and 3-D bubble rising/drop falling flow [20].

REFERENCES